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Semi-invariant Riemannian maps from almost Hermitian manifolds

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Abstract

We construct Gauss–Weingarten-like formulas and define O'Neill's tensors for Riemannian maps between Riemannian manifolds. By using these new formulas, we obtain necessary and sufficient conditions for Riemannian maps to be totally geodesic. Then we introduce semi-invariant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, give examples and investigate the geometry of leaves of the distributions defined by such maps. We also obtain necessary and sufficient conditions for semi-invariant maps to be totally geodesic and find decomposition theorems for the total manifold. Finally, we give a classification result for semi-invariant Riemannian maps with totally umbilical fibers.

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1. Introduction

In differential geometry, it is desirable to introduce and use suitable types of maps between Riemannian (or semi-Riemannian) manifolds. Such maps may help to compare geometric properties of manifolds. Isometric immersions and Riemannian submersions are used widely in differential geometry as differential maps between Riemannian manifolds to compare geometric structures defined on both manifolds. We recall that a smooth map $F : (M_1, g_1) \longrightarrow (M_2, g_2)$

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between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called an isometric immersion (submanifold) if F_* is injective and

$$g_2(F_*X, F_*Y) = g_1(X, Y) \quad (1.1)$$

for X, Y vector fields tangent to M_1 , here F_* denotes the derivative map.

It is known that the complex techniques in relativity have been very effective tools for understanding spacetime geometry [15]. Indeed, complex manifolds have two interesting classes of Kähler manifolds. One is Calabi–Yau manifolds which have their applications in superstring theory [4]. The other one is Teichmüller spaces applicable to relativity [26]. It is also important to note that CR-structures have been extensively used in spacetime geometry of relativity [20]. For complex methods in general relativity, see: [8].

The theory of submanifolds of Kähler manifolds is one of the important branches of differential geometry. A submanifold of a Kähler manifold is a complex (invariant) submanifold if the tangent space of the submanifold at each point is invariant with respect to the almost complex structure of the Kähler manifold. Besides complex submanifolds of a Kähler manifold, there is another important class of submanifolds called totally real submanifolds. A totally real submanifold of a Kähler manifold \bar{M} is a submanifold of \bar{M} such that the almost complex structure J of \bar{M} carries the tangent space of the submanifold at each point into its normal space and the main properties of such submanifolds established in [5,18,28]. On the other hand, CR-submanifolds were defined by Bejancu [3] as a generalization of complex and totally real submanifolds. A CR-submanifold is called proper if it is neither complex nor totally real submanifold. The geometry of CR-submanifolds has been studied in several papers since then.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds of dimension m_1 and n_1 , respectively. A smooth map $F : (M_1, g_1) \rightarrow (M_2, g_2)$ is called a Riemannian submersion if F_* is onto and it satisfies the Eq. (1.1) for vector fields tangent to the horizontal space $(\ker F_*)^\perp$. Riemannian submersions between Riemannian manifolds were studied by O’Neill [19] and Gray [12], see also [9]. Later such submersions were considered between manifolds with differentiable structures. First note that a smooth map $\phi : M \rightarrow N$ between almost complex manifolds (M, J) and (N, \bar{J}) is called almost complex (or holomorphic) map if $\phi_*(JX) = \bar{J}\phi_*(X)$ for $X \in \Gamma(TM)$, where J and \bar{J} are complex structures of M and N , respectively. As an analogue of holomorphic submanifolds and almost complex maps, Watson [27] defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases. We note that almost Hermitian submersions have been extended to the almost contact manifolds [6], locally conformal Kähler manifolds [16] and quaternion Kähler manifolds [14]. We also note that Riemannian submersions have their applications in spacetime of unified theory. In the theory of Kaluza–Klein type, a general solution of the non-linear sigma model is given by Riemannian submersions from the extra dimensional space to the space in which the scalar fields of the nonlinear sigma model take values, for details see: [9].

But one can observe that the isometric immersions and Riemannian submersions are very special maps between Riemannian manifolds by considering arbitrary differentiable maps. In other words, an arbitrary map between Riemannian manifolds may not be isometric immersion or Riemannian submersion. In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [10] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F : (M_1, g_1) \rightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \text{rank } F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. Then we denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker F_*)^\perp$

to $\ker F_*$. Then the tangent bundle of M_1 has the following decomposition

$$TM_1 = \ker F_* \oplus \mathcal{H}.$$

We denote the range of F_* by $\text{range } F_*$ and consider the orthogonal complementary space $(\text{range } F_*)^\perp$ to $\text{range } F_*$ in the tangent bundle TM_2 of M_2 . Since $\text{rank } F < \min\{m, n\}$, we always have $(\text{range } F_*)^\perp \neq \{0\}$. Thus the tangent bundle TM_2 of M_2 has the following decomposition

$$TM_2 = (\text{range } F_*) \oplus (\text{range } F_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \rightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F_{*p_1}^h : (\ker F_{*p_1})^\perp \rightarrow (\text{range } F_{*p_1})$ is a linear isometry between the inner product spaces $((\ker F_{*p_1})^\perp, g_1(p_1)|_{(\ker F_{*p_1})^\perp})$ and $(\text{range } F_{*p_1}, g_2(p_2)|_{(\text{range } F_{*p_1})})$, $p_2 = F(p_1)$. Therefore Fischer stated in [10] that a Riemannian map is a map which is as isometric as it can be. In another words, F_* satisfies the Eq. (1.1) for X, Y vector fields tangent to \mathcal{H} . It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\text{range } F_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion which implies that the rank of the linear map $F_{*p} : T_p M_1 \rightarrow T_{F(p)} M_2$ is constant for p in each connected component of M_1 , [1,10]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see: [11].

In [25], we introduced semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and anti-invariant submersions [22], then we studied the geometry of such maps. We recall that a Riemannian submersion F from an almost Hermitian manifold (M, J_M, g_M) with almost complex structure J_M to a Riemannian manifold (N, g_N) is called a semi-invariant submersion if the fibers have differentiable distributions D and D^\perp such that D is invariant with respect to J_M and its orthogonal complement D^\perp is totally real distribution, i.e., $J_M(D_p^\perp) \subseteq (\ker F_{*p})^\perp$. Obviously, almost Hermitian submersions and anti-invariant submersions are semi-invariant submersions with $D^\perp = \{0\}$ and $D = \{0\}$, respectively.

In this paper, we first construct Gauss–Weingarten-like formulas and introduce O’Neill’s type tensor fields for Riemannian maps. We obtain new conditions for Riemannian maps to be totally geodesic in terms of these notions. Then we introduce semi-invariant Riemannian maps from almost Hermitian manifolds, give examples and study the geometry of such maps.

The paper is organized as follows: In Section 2, we present the basic background needed for this paper. In Section 3, we study Riemannian maps and obtain necessary and sufficient conditions for Riemannian maps to be totally geodesic. In Section 4, we define semi-invariant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, give examples and obtain integrability conditions for distributions defined by semi-invariant Riemannian maps. We also find necessary and sufficient conditions for semi-invariant maps to be totally geodesic. In the last section of this paper, we study semi-invariant Riemannian maps with totally umbilical fibers and obtain classification theorem for such Riemannian maps.

An important initial step in understanding a complicated mathematical object is to decompose it into simpler irreducible components. In differential geometry, a fundamental result in this direction is the decomposition theorem of de Rham, which gives necessary and sufficient conditions for a Riemannian manifold to split, both locally and globally, into a Riemannian product of Riemannian manifolds [7]. It is known that decomposition theorems have been widely used in mathematical physics. Indeed, many spacetime models of general relativity are examples

of product manifolds, warped product manifolds [13] or twisted product manifolds [21]. In the Section 4 of the paper, we also obtain decomposition theorems for the total manifold by using semi-invariant Riemannian maps.

2. Preliminaries

In this section we recall some basic materials from [2,29]. Let (M, g_M) be a Riemannian manifold and \mathcal{V} be a q -dimensional distribution on M . Denote its orthogonal distribution \mathcal{V}^\perp by \mathcal{H} . Then, we have

$$TM = \mathcal{V} \oplus \mathcal{H}. \quad (2.1)$$

\mathcal{V} is called the vertical distribution and \mathcal{H} is called the horizontal distribution. We use the same letters to denote the orthogonal projections onto these distributions.

By the unsymmetrized second fundamental form of \mathcal{V} , we mean the tensor field $A^\mathcal{V}$ defined by

$$A_E^\mathcal{V} F = \mathcal{H}(\nabla_{\mathcal{V}E} \mathcal{V}F), \quad E, F \in \Gamma(TM), \quad (2.2)$$

where ∇ is the Levi-Civita connection on M . The symmetrized second fundamental form $B^\mathcal{V}$ of \mathcal{V} is given by

$$B^\mathcal{V}(E, F) = \frac{1}{2}\{A_E^\mathcal{V} F + A_F^\mathcal{V} E\} = \frac{1}{2}\{\mathcal{H}(\nabla_{\mathcal{V}E} \mathcal{V}F) + \mathcal{H}(\nabla_{\mathcal{V}F} \mathcal{V}E)\} \quad (2.3)$$

for any $E, F \in \Gamma(TM)$. The integrability tensor of \mathcal{V} is the tensor field $I^\mathcal{V}$ given by

$$I^\mathcal{V}(E, F) = A_E^\mathcal{V} F - A_F^\mathcal{V} E - \mathcal{H}([\mathcal{V}E, \mathcal{V}F]). \quad (2.4)$$

Moreover, the mean curvature of \mathcal{V} is defined by

$$\mu^\mathcal{V} = \frac{1}{q} \text{Trace } B^\mathcal{V} = \frac{1}{q} \sum_{i=1}^q \mathcal{H}(\nabla_{e_i} e_i), \quad (2.5)$$

where $\{e_1, \dots, e_q\}$ is a local frame of \mathcal{V} . By reversing the roles of \mathcal{V} , \mathcal{H} , $B^\mathcal{H}$, $A^\mathcal{H}$ and $I^\mathcal{H}$ can be defined similarly. For instance, $B^\mathcal{H}$ is defined by

$$B^\mathcal{H}(E, F) = \frac{1}{2}\{\mathcal{V}(\nabla_{\mathcal{H}E} \mathcal{H}F) + \mathcal{V}(\nabla_{\mathcal{H}F} \mathcal{H}E)\} \quad (2.6)$$

and, hence we have

$$\mu^\mathcal{H} = \frac{1}{m-q} \text{Trace } B^\mathcal{H} = \frac{1}{m-q} \sum_{s=1}^{m-q} \mathcal{V}(\nabla_{E_s} E_s), \quad (2.7)$$

where E_1, \dots, E_{m-q} is a local frame of \mathcal{H} . A distribution \mathcal{D} on M is said to be minimal if, for each $x \in M$, the mean curvature vanishes.

A $2k$ -dimensional Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ is called an almost Hermitian manifold if there exists a tensor field \bar{J} of type $(1, 1)$ on \bar{M} such that $\bar{J}^2 = -I$ and

$$\bar{g}(X, Y) = \bar{g}(\bar{J}X, \bar{J}Y), \quad \forall X, Y \in \Gamma(T\bar{M}), \quad (2.8)$$

where I denotes the identity transformation of $T_p\bar{M}$. Consider an almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ and denote by $\bar{\nabla}$ the Levi-Civita connection on \bar{M} with respect to \bar{g} . Then \bar{M} is called a Kähler manifold [29] if \bar{J} is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X \bar{J})Y = 0 \quad (2.9)$$

for $X, Y \in \Gamma(T\bar{M})$.

3. Riemannian maps

In this section, we develop fundamental formulas for Riemannian maps similar to the Gauss–Weingarten formulas of isometric immersions and O’Neill’s formulas of Riemannian submersions. We also recall useful results which are related to the second fundamental form and the tension field of the Riemannian maps. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \rightarrow N$ is a smooth map between them. Then the differential F_* of F can be viewed a section of the bundle $\text{Hom}(TM, F^{-1}TN) \rightarrow M$, where $F^{-1}TN$ is the pullback bundle which has fibers $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. $\text{Hom}(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X^M Y) \quad (3.1)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. First note that in [24] we showed that the second fundamental form $(\nabla F_*)(X, Y)$, $\forall X, Y \in \Gamma((\ker F_*)^\perp)$, of a Riemannian map has no components in range F_* . More precisely we have the following.

Lemma 3.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \quad \forall X, Y, Z \in \Gamma((\ker F_*)^\perp).$$

As a result of Lemma 3.1, we have

$$(\nabla F_*)(X, Y) \in \Gamma((\text{range } F_*)^\perp), \quad \forall X, Y \in \Gamma((\ker F_*)^\perp). \quad (3.2)$$

For the tension field of a Riemannian map between Riemannian manifolds, we have the following.

Lemma 3.2 ([23]). *Let $F : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then the tension field τ of F is*

$$\tau = -m_1 F_*(H) + m_2 H_2, \quad (3.3)$$

where $m_1 = \dim((\ker F_*))$, $m_2 = \text{rank } F$, H and H_2 are the mean curvature vector fields of the distribution $\ker F_*$ and range F_* , respectively.

Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then we define \mathcal{T} and \mathcal{A} as

$$\mathcal{A}_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H} F \quad (3.4)$$

$$\mathcal{T}_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H} F, \quad (3.5)$$

for vector fields E, F on M_1 , where ∇ is the Levi-Civita connection of g_1 . In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM_1)$, \mathcal{T}_E and \mathcal{A}_E are skew-symmetric operators on $(\Gamma(TM_1), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that \mathcal{T} is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and \mathcal{A} is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field \mathcal{T} satisfies

$$\mathcal{T}_U W = \mathcal{T}_W U, \quad \forall U, W \in \Gamma(\ker F_*). \quad (3.6)$$

On the other hand, from (3.4) and (3.5) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \quad (3.7)$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \quad (3.8)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \quad (3.9)$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \quad (3.10)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F . Then according to [17], for any vector field X on M_1 and any section V of $(\text{range } F_*)^\perp$, where $(\text{range } F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fiber $(F_*(T_p M))^\perp$ -orthogonal complement of $F_*(T_p M)$ for g_2 over p , we have $\nabla_X^{F\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. In [17], the author also showed that $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_2 = 0$. We now define \mathcal{S}_V as

$$\nabla_{F_*X}^2 V = -\mathcal{S}_V F_*X + \nabla_X^{F\perp} V, \quad (3.11)$$

where $\mathcal{S}_V F_*X$ is the tangential component (a vector field along F) of $\nabla_{F_*X}^2 V$. It is easy to see that $\mathcal{S}_V F_*X$ is bilinear in V and F_*X and $\mathcal{S}_V F_*X$ at p depends only on V_p and $F_{*p}X_p$. By direct computations, we obtain

$$g_2(\mathcal{S}_V F_*X, F_*Y) = g_2(V, (\nabla F_*)(X, Y)), \quad (3.12)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that \mathcal{S}_V is a symmetric linear transformation of $\text{range } F_*$.

We now give necessary and sufficient conditions for a Riemannian map to be totally geodesic in terms of (3.4), (3.5) and (3.11). We recall that a differentiable map F between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 3.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then F is totally geodesic if and only if*

- (a) $\mathcal{A}_X Y = 0$,
- (b) $\mathcal{S}_V F_*(X) = 0$,
- (c) *the fibers are totally geodesic,*

for $X, Y \in \Gamma((\ker f_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$.

Proof. First note that a map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) is totally geodesic if and only if $(\nabla F_*)(X, Y) = 0$, $(\nabla F_*)(X, U) = 0$ and $(\nabla F_*)(U_1, U_2) = 0$ for $U, U_1, U_2 \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$. Since $(\nabla F_*)(X, U) =$

$\Gamma(\text{range } F_*)$, $(\nabla F_*)(X, U) = 0$ if and only if $g_2((\nabla F_*)(X, U), F_*(Y)) = 0$ for $Y \in \Gamma((\ker F_*)^\perp)$. By using (3.1) and (3.4) we have

$$g_1(\mathcal{A}_X U, Y) = -g_2((\nabla F_*)(X, U), F_*(Y)). \quad (3.13)$$

In a similar way, since $(\nabla F_*)(U, V) \in \Gamma(\text{range } F_*)$, it follows that $(\nabla F_*)(U, V) = 0$ if and only if $g_2((\nabla F_*)(U, V), F_*(X)) = 0$ for $X \in \Gamma((\ker F_*)^\perp)$. Then from (3.1) and (3.5) we get

$$g_1(\mathcal{T}_U V, X) = -g_2((\nabla F_*)(U, V), F_*(X)). \quad (3.14)$$

On the other hand, for $X, Y \in \Gamma((\ker F_*)^\perp)$, since $(\nabla F_*)(X, Y) \in \Gamma((\text{range } F_*)^\perp)$, it follows that $(\nabla F_*)(X, Y) = 0$ if and only if $g_2((\nabla F_*)(X, Y), V) = 0$ for $V \in \Gamma((\text{range } F_*)^\perp)$. Then using (3.12), we obtain

$$g_2((\nabla F_*)(X, Y), V) = g_2(F_*(Y), S_V F_*(X)). \quad (3.15)$$

Thus the assertion comes from (3.8), (3.13) and (3.15). \square

4. Semi-invariant Riemannian maps

In this section we introduce semi-invariant Riemannian maps from almost Hermitian manifolds, give examples, investigate the geometry of leaves of the distributions defined by semi-invariant Riemannian maps and obtain decomposition theorems.

Definition 4.1. Let F be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (M_2, g_N) . Then we say that F is a semi-invariant Riemannian map if the following conditions are satisfied;

(A) There exist a subbundle of $\ker F_*$ such that

$$J(D_1) = D_1.$$

(B) There exist a complementary subbundle D_2 to D_1 in $\ker F_*$ such that

$$J(D_2) \subseteq (\ker F_*)^\perp.$$

From definition, we have

$$\ker F_* = D_1 \oplus D_2. \quad (4.1)$$

Now, we denote the orthogonal complementary subbundle of $(\ker F_*)^\perp$ to $J(D_2)$ by μ . Then it is easy to see that μ is invariant. We now provide some examples of semi-invariant Riemannian maps.

Example 4.1. Every holomorphic submersion between almost Hermitian manifolds is a semi-invariant Riemannian map with $D_1 = \ker F_*$ and $(\text{range } F_*)^\perp = \{0\}$.

Example 4.2. Every anti-invariant Riemannian submersion from an almost Hermitian manifold to a Riemannian manifold is a semi-invariant Riemannian map $D_2 = \ker F_*$ and $(\text{range } F_*)^\perp = \{0\}$.

Example 4.3. Every semi-invariant submersion from an almost hermitian manifold to a Riemannian manifold is a semi-invariant Riemannian map with $(\text{range } F_*)^\perp = \{0\}$.

In the following R^{2m} denotes the Euclidean $2m$ -space with the standard metric. An almost complex structure J on R^{2m} is said to be compatible if (R^{2m}, J) is complex analytically

isometric to the complex number space C^m with the standard flat Kählerian metric. We denote by J the compatible almost complex structure on R^{2m} defined by

$$J(a^1, \dots, a^{2m}) = (-a^2, a^1, \dots, -a^{2m}, a^{2m-1}).$$

We say that a semi-invariant Riemannian map is proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\mu \neq \{0\}$. Here is an example of a proper semi-invariant Riemannian map.

Example 4.4. Consider the following map defined by

$$F : \quad R^6 \quad \longrightarrow \quad R^4 \\ (x^1, x^2, x^3, x^4, x^5, x^6) \quad \left(\frac{x^1 - x^3}{\sqrt{2}}, \frac{x^2 - x^4}{\sqrt{2}}, \frac{x^5 + x^6}{\sqrt{2}}, 0 \right).$$

Then we have

$$\ker F_* = \text{span} \left\{ Z_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}, Z_2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4}, Z_3 = \frac{\partial}{\partial x^5} - \frac{\partial}{\partial x^6} \right\}$$

and

$$(\ker F_*)^\perp = \text{span} \left\{ Z_4 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3}, Z_5 = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, Z_6 = \frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^6} \right\}.$$

Hence it is easy to see that

$$g_{R^4}(F_*(Z_i), F_*(Z_i)) = g_{R^6}(Z_i, Z_i) = 2$$

and

$$g_{R^4}(F_*(Z_i), F_*(Z_j)) = g_{R^6}(Z_i, Z_j) = 0,$$

$i \neq j$, for $i, j = 4, 5, 6$. Thus F is a Riemannian map. On the other hand, we have $JZ_1 = Z_2$ and $JZ_3 = Z_6 \in \Gamma((\ker F_*)^\perp)$, where J is the complex structure of R^6 . Thus F is a semi-invariant Riemannian map with $D_1 = \text{span}\{Z_1, Z_2\}$, $D_2 = \text{span}\{Z_3\}$ and $\mu = \text{span}\{Z_4, Z_5\}$.

Let F be a semi-invariant Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then for $U \in \Gamma(\ker F_*)$, we write

$$JU = \phi U + \omega U, \quad (4.2)$$

where $\phi U \in \Gamma(D_1)$ and $\omega U \in \Gamma(JD_2)$. Also for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$JX = \mathcal{B}X + \mathcal{C}X, \quad (4.3)$$

where $\mathcal{B}X \in \Gamma(D_2)$ and $\mathcal{C}X \in \Gamma(\mu)$.

Since F is a subimmersion, it follows that the rank of F is constant on M_1 , then the rank theorem for functions implies that $\ker F_*$ is an integrable subbundle of TM_1 , [1, page: 205]. For the integral manifolds of $\ker F_*$, we have the following.

Theorem 4.1. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $\ker F_*$ defines a totally geodesic foliation if and only if*

$$g_N((\nabla F_*)(U, \omega V), F_*(\mathcal{C}X)) = g_M(\hat{\nabla}_U \phi V, \mathcal{B}X) + g_M(\mathcal{T}_U \phi V, \mathcal{C}X) - g_M(\omega V, \mathcal{T}_U \mathcal{B}X)$$

for $U, V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. It is clear that $\ker F_*$ defines a totally geodesic foliation if and only if $g_M(\nabla_U V, X) = 0$ for $U, V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$. Using (4.2), (4.3) and (3.7) we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\hat{\nabla}_U \phi V, \mathcal{B}X) + g_M(\mathcal{T}_U \phi V, \mathcal{C}X) \\ &\quad - g_M(\omega V, \mathcal{T}_U \mathcal{B}X) + g_M(\nabla_U \omega V, \mathcal{C}X) \end{aligned}$$

for $U, V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$. Since F is a Riemannian map, by using (3.1) we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\hat{\nabla}_U \phi V, \mathcal{B}X) + g_M(\mathcal{T}_U \phi V, \mathcal{C}X) \\ &\quad - g_M(\omega V, \mathcal{T}_U \mathcal{B}X) - g_N((\nabla_U F_*)(U, \omega V), F_*(\mathcal{C}X)) \end{aligned}$$

which proves the assertion. \square

For the integrability of the distribution $(\ker F_*)^\perp$, we have the following result.

Theorem 4.2. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ is integrable if and only if*

$$\begin{aligned} g_M(\mathcal{B}Z_2, \mathcal{A}_{Z_1} \omega U) - g_M(\mathcal{B}Z_1, \mathcal{A}_{Z_2} \omega U) \\ = g_M(\nabla_{Z_1} \mathcal{B}Z_2 - \nabla_{Z_2} \mathcal{B}Z_1, \phi U) + g_N((\nabla F_*)(Z_1, \phi U) - \nabla_{Z_1}^F F_*(\omega U), F_*(\mathcal{C}Z_2)) \\ - g_N((\nabla F_*)(Z_2, \phi U) - \nabla_{Z_2}^F F_*(\omega U), F_*(\mathcal{C}Z_1)) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. By direct computations, we have

$$g_M([Z_1, Z_2], U) = -g_M(Z_2, \nabla_{Z_1} U) + g_M(Z_1, \nabla_{Z_2} U)$$

for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. Then using (2.8), (2.9), (4.2), (4.3) and (3.10) we get

$$\begin{aligned} g_M([Z_1, Z_2], U) &= -g_M(\mathcal{B}Z_2, \nabla_{Z_1} \phi U) - g_M(\mathcal{B}Z_2, \mathcal{A}_{Z_1} \omega U) - g_M(\mathcal{C}Z_2, \nabla_{Z_1} \phi U) \\ &\quad - g_M(\mathcal{C}Z_2, \nabla_{Z_1} \omega U) + g_M(\mathcal{B}Z_1, \nabla_{Z_2} \phi U) + g_M(\mathcal{B}Z_1, \mathcal{A}_{Z_2} \omega U) \\ &\quad + g_M(\mathcal{C}Z_1, \nabla_{Z_2} \phi U) + g_M(\mathcal{C}Z_1, \nabla_{Z_2} \omega U). \end{aligned}$$

Since F is a Riemannian map, from (3.1) and (3.2) we obtain

$$\begin{aligned} g_M([Z_1, Z_2], U) &= -g_M(\mathcal{B}Z_2, \nabla_{Z_1} \phi U) - g_M(\mathcal{B}Z_2, \mathcal{A}_{Z_1} \omega U) \\ &\quad + g_N(F_*(\mathcal{C}Z_2), (\nabla F_*)(Z_1, \phi U)) - g_N(F_*(\mathcal{C}Z_2), \nabla_{Z_1}^F F_*(\omega U)) \\ &\quad + g_M(\mathcal{B}Z_1, \nabla_{Z_2} \phi U) + g_M(\mathcal{B}Z_1, \mathcal{A}_{Z_2} \omega U) \\ &\quad - g_N(F_*(\mathcal{C}Z_1), (\nabla F_*)(Z_2, \phi U)) + g_N(F_*(\mathcal{C}Z_1), \nabla_{Z_2}^F F_*(\omega U)) \end{aligned}$$

which gives the assertion. \square

For the leaves of the distribution $(\ker F_*)^\perp$, we have the following result.

Theorem 4.3. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if*

$$g_N(\nabla_X^F F_*(CY), F_*(\omega U)) = g_M(\mathcal{A}_X \phi U, CY) - g_M(\mathcal{A}_X BY, \omega U) - g_M(\nabla_X BY, \phi U)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. From (2.8), (2.9), (3.9), (4.2) and (4.3) we obtain

$$g_M(\nabla_X Y, U) = g_M(\nabla_X \mathcal{B}Y, \phi U) + g_M(\mathcal{A}_X \mathcal{B}Y, \omega U) \\ - g_M(\mathcal{C}Y, \mathcal{A}_X \phi U) + g_M(\nabla_X \mathcal{C}Y, \omega U)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. Since F is a semi-invariant Riemannian map, using (3.1) we get the assertion. \square

From Theorems 4.1 and 4.3, we have the following decomposition theorem.

Theorem 4.4. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then M is a locally product Riemannian manifold if and only if*

$$g_N(\nabla_X^F F_*(CY), F_*(\omega U)) = g_M(\mathcal{A}_X \phi U, CY) - g_M(\mathcal{A}_X \mathcal{B}Y, \omega U) - g_M(\nabla_X \mathcal{B}Y, \phi U)$$

and

$$g_N((\nabla F_*)(U, \omega V), F_*(CX)) = g_M(\hat{\nabla}_U \phi V, \mathcal{B}X) + g_M(\mathcal{T}_U \phi V, \mathcal{C}X) - g_M(\omega V, \mathcal{T}_U \mathcal{B}X)$$

for $U, V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Since $(\ker F_*)^\perp = J(D_2) \oplus \mu$ and F is a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) , for $X \in \Gamma(D_2)$ and $Y \in \Gamma(\mu)$, we have

$$g_N(F_*(JX), F_*(Y)) = g_M(JX, Y) = 0.$$

This implies that the distributions $F_*(JD_2)$ and $F_*(\mu)$ are orthogonal. Thus, if we denote $F_*(JD_2)$ and $F_*(\mu)$ by \bar{D}_2 and $\bar{\mu}$, we have the following decomposition for $(\text{range } F_*)$

$$\text{range } F_* = \bar{D}_2 \oplus \bar{\mu}. \quad (4.4)$$

We investigate the geometry of the leaves of the distributions D_1 and D_2 .

Theorem 4.5. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then D_1 defines a totally geodesic foliation on M if and only if*

$$(\nabla F_*)(X_1, JY_1) \in \Gamma(\bar{\mu})$$

and

$$g_M(\hat{\nabla}_{X_1} JY_1, \mathcal{B}X) = g_N((\nabla F_*)(X_1, JY_1), F_*(CX))$$

for $X_1, Y_1 \in \Gamma(D_1)$, $X_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. From the definition of a semi-invariant Riemannian map, it follows that the distribution D_1 defines a totally geodesic foliation on M if and only if $g_M(\nabla_{X_1} Y_1, X_2) = 0$ and $g_M(\nabla_{X_1} Y_1, X) = 0$ for $X_1, Y_1 \in \Gamma(D_1)$, $X_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$. Since F is a Riemannian map, from (3.1) we have

$$g_M(\nabla_{X_1} Y_1, X_2) = -g_N((\nabla F_*)(X_1, JY_1), F_*(JX_2)). \quad (4.5)$$

On the other hand, by using (4.3) we derive

$$g_M(\nabla_{X_1} Y_1, X) = g_M(\nabla_{X_1} JY_1, \mathcal{B}X) + g_M(\nabla_{X_1} JY_1, \mathcal{C}X).$$

Then Riemannian map F , (3.1) and (3.7) imply that

$$g_M(\nabla_{X_1} Y_1, X) = g_M(\hat{\nabla}_{X_1} JY_1, \mathcal{B}X) - g_N((\nabla F_*)(X_1, JY_1), F_*(\mathcal{C}X)). \quad (4.6)$$

Thus proof follows from (4.4)–(4.6). \square

For the leaves of D_2 we have the following result.

Theorem 4.6. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then D_2 defines a totally geodesic foliation on M if and only if*

$$(\nabla F_*)(X_2, JY_1) \in \Gamma(\bar{\mu})$$

and

$$g_M(\mathcal{T}_{X_2} \mathcal{B}X, JY_2) = -g_N((\nabla F_*)(X_2, JY_2), F_*(\mathcal{C}X))$$

for $X_1 \in \Gamma(D_1)$, $X_2, Y_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. From the definition of a semi-invariant Riemannian map, it follows that the distribution D_2 defines a totally geodesic foliation on M if and only if $g_M(\nabla_{X_2} Y_2, X_1) = 0$ and $g_M(\nabla_{X_2} Y_2, X) = 0$ for $X_1 \in \Gamma(D_1)$, $X_2, Y_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$. Since we have $g_M(\nabla_{X_2} Y_2, X_1) = -g_M(\nabla_{X_2} X_1, Y_2)$, from (3.1) we have

$$g_M(\nabla_{X_2} Y_2, X_1) = g_N((\nabla F_*)(X_2, JX_1), F_*(JY_2)). \quad (4.7)$$

In a similar way, by using (4.3) we derive

$$g_M(\nabla_{X_2} Y_2, X) = -g_M(\nabla_{X_2} \mathcal{B}X, JY_2) + g_M(\nabla_{X_2} JY_2, \mathcal{C}X).$$

Then Riemannian map F , (3.1) and (3.7) imply that

$$g_M(\nabla_{X_1} Y_1, X) = -g_M(JY_2, \mathcal{T}_{X_2} \mathcal{B}X) - g_N((\nabla F_*)(X_2, JY_2), F_*(\mathcal{C}X)). \quad (4.8)$$

The proof follows from (4.4), (4.7) and (4.8). \square

From (4.5) and (4.7) we have the following result.

Corollary 4.1. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the fibers are locally product Riemannian manifolds if and only if*

$$(\nabla F_*)(U, JY_1) \in \Gamma(\bar{\mu})$$

for $U \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(D_1)$.

Let $F : (M, g_M) \longrightarrow (N, g_N)$ be a map between Riemannian manifolds (M, g_M) and (N, g_N) . Then the adjoint map *F_* of F_* is characterized by $g_M(x, {}^*F_{*p_1} y) = g_N(F_{*p_1} x, y)$ for $x \in T_{p_1} M$, $y \in T_{F(p_1)} N$ and $p_1 \in M$. Considering F_*^h at each $p_1 \in M$ as a linear transformation

$$F_{*p_1}^h : ((\ker F_*)^\perp(p_1), g_M|_{(\ker F_*)^\perp(p_1)}) \longrightarrow (\text{range } F_*(p_2), g_N|_{(\text{range } F_*)(p_2)}),$$

we will denote the adjoint of F_*^h by ${}^*F_{*p_1}^h$. Let ${}^*F_{*p_1}$ be the adjoint of $F_{*p_1} : (T_{p_1} M, g_M|_{p_1}) \longrightarrow (T_{p_2} N, g_N|_{p_2})$. Then the linear transformation

$$({}^*F_{*p_1})^h : \text{range } F_*(p_2) \longrightarrow (\ker F_*)^\perp(p_1)$$

defined by $(*F_{*p_1})^h y = *F_{*p_1} y$, where $y \in \Gamma(\text{range } F_{*p_1})$, $p_2 = F(p_1)$, is an isomorphism and $(F_{*p_1}^h)^{-1} = (*F_{*p_1})^h = *(F_{*p_1}^h)$.

Finally, in this section, we give a characterization for semi-invariant Riemannian maps to be totally geodesic in terms of \mathcal{A} , \mathcal{T} and \mathcal{S} .

Theorem 4.7. *Let F be a semi-invariant Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then F is totally geodesic if and only if*

$$\begin{aligned} g_M(\hat{\nabla}_X \mathcal{B}Z_1, \mathcal{B}Z_2) &= g_M(X, \mathcal{A}_{CZ_1} C Z_2) - g_M(\mathcal{T}_X \mathcal{B}Z_1, C Z_2) - g_M(\mathcal{T}_X C Z_1, \mathcal{B}Z_2), \\ g_M(\hat{\nabla}_X \mathcal{B}Z, \phi Y) &= -\{g_M(\phi Y, \mathcal{T}_X C Z) + g_M(\omega Y, \mathcal{T}_X \mathcal{B}Z) + g_M(\mathcal{A}_{\omega Y} C Z, X)\} \end{aligned}$$

and

$$*F_*(\mathcal{S}_V F_*(JU)) = 0, \quad *F_*(\mathcal{S}_V F_*(Z_3)) \in \Gamma(J(D_2))$$

for $Z_1, Z_2, Z \in \Gamma((\ker F_*)^\perp)$, $X, Y \in \Gamma(\ker F_*)$, $U \in \Gamma(D_2)$, $Z_3 \in \Gamma(\mu)$ and $V \in \Gamma((\text{range } F_*)^\perp)$.

Proof. From the definition of a Riemannian map, F is totally geodesic if and only if $(\nabla F_*)(X, Y) = 0$, $(\nabla F_*)(X, Z) = 0$ and $(\nabla F_*)(Z_1, Z_2) = 0$ for $X, Y \in \Gamma(\ker F_*)$, $Z, Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$. From the above information, decomposition of the tangent bundle of M and (3.2), a semi-invariant Riemannian map F is totally geodesic if and only if

$$g_N((\nabla F_*)(X, Z_1), F_*(Z_2)) = 0, \quad g_N((\nabla F_*)(X, Y), F_*(Z)) = 0$$

and

$$g_N((\nabla F_*)(JU, Z), V) = 0, \quad g_N((\nabla F_*)(Z_3, Z_4), V) = 0$$

for $X, Y \in \Gamma(\ker F_*)$, $U \in \Gamma(D_2)$, $Z, Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, $Z_3, Z_4 \in \Gamma(\mu)$. Since F is a Riemannian map, by using (3.1), (2.8), (2.9) and (4.3) we have

$$\begin{aligned} g_N((\nabla F_*)(X, Z_1), F_*(Z_2)) &= -g_M(\nabla_X \mathcal{B}Z_1, \mathcal{B}Z_2) - g_M(\nabla_X C Z_1, \mathcal{B}Z_2) \\ &\quad - g_M(\nabla_X \mathcal{B}Z_1, C Z_2) - g_M(\nabla_X C Z_1, C Z_2). \end{aligned}$$

Then from (3.7), (3.8) and (3.1) we get

$$\begin{aligned} g_N((\nabla F_*)(X, Z_1), F_*(Z_2)) &= -g_M(\hat{\nabla}_X \mathcal{B}Z_1, \mathcal{B}Z_2) - g_M(\mathcal{T}_X C Z_1, \mathcal{B}Z_2) \\ &\quad - g_M(\mathcal{T}_X \mathcal{B}Z_1, C Z_2) + g_N((\nabla F_*)(X, C Z_1), F_*(C Z_2)). \end{aligned} \quad (4.9)$$

On the other hand, since the second fundamental form is symmetric, we get

$$g_N((\nabla F_*)(X, C Z_1), F_*(C Z_2)) = g_M(X, \nabla_{C Z_1} C Z_2).$$

Thus from (3.10) we obtain

$$g_N((\nabla F_*)(X, C Z_1), F_*(C Z_2)) = g_M(X, \mathcal{A}_{C Z_1} C Z_2). \quad (4.10)$$

Using (4.10) in (4.9) we have

$$\begin{aligned} g_N((\nabla F_*)(X, Z_1), F_*(Z_2)) &= -g_M(\hat{\nabla}_X \mathcal{B}Z_1, \mathcal{B}Z_2) - g_M(\mathcal{T}_X C Z_1, \mathcal{B}Z_2) \\ &\quad - g_M(\mathcal{T}_X \mathcal{B}Z_1, C Z_2) + g_M(X, \mathcal{A}_{C Z_1} C Z_2). \end{aligned} \quad (4.11)$$

In a similar way, we get

$$\begin{aligned} g_N((\nabla F_*)(X, Y), F_*(Z)) &= g_M(\hat{\nabla}_X \mathcal{B}Z, \phi Y) + g_M(\mathcal{T}_X \mathcal{C}Z, \phi Y) \\ &\quad + g_M(\mathcal{T}_X \mathcal{B}Z, \omega Y) + g_M(\mathcal{A}_{\omega Y} \mathcal{C}Z, X). \end{aligned} \quad (4.12)$$

On the other hand, from (3.12) we have

$$g_N((\nabla F_*)(JU, Z), V) = g_N(F_*(Z), \mathcal{S}_V F_*(JU)).$$

Then using the adjoint map, we obtain

$$g_N((\nabla F_*)(JU, Z), V) = g_M(Z, {}^*F_* \mathcal{S}_V F_*(JU)). \quad (4.13)$$

In a similar way, we have

$$g_N((\nabla F_*)(Z_3, Z_4), V) = g_M(Z_4, {}^*F_* \mathcal{S}_V F_*(Z_3)). \quad (4.14)$$

Thus proof follows from (4.11)–(4.14). \square

5. Semi-invariant Riemannian maps with totally umbilical fibers

In this section, we study semi-invariant Riemannian maps with totally umbilical fibers and obtain a characterization for such semi-invariant Riemannian maps. We first define semi-invariant Riemannian maps with totally umbilical fibers by using the definition given for a Riemannian submersion. More precisely, a semi-invariant Riemannian map F from an almost Hermitian manifold (M, J, g_M) to a Riemannian manifold (N, g_N) is called a semi-invariant Riemannian map with totally umbilical fibers if

$$\mathcal{T}_U V = g_M(U, V)H \quad (5.1)$$

for $U, V \in \Gamma(\ker F_*)$, where H is the mean curvature vector field of $\ker F_*$.

Lemma 5.1. *Let F be a semi-invariant Riemannian map F from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) with totally umbilical fibers. Then $H \in \Gamma(JD_2)$.*

Proof. Since M is a Kähler manifold, we have $\nabla_X JY = J\nabla_X Y$ for $X, Y \in \Gamma(D_1)$. Then using (3.7), (4.2), (4.3) and (5.1), for $W \in \Gamma(\mu)$ we get

$$g_M(X, JY)g_M(H, W) = -g_M(X, Y)g_M(H, JW). \quad (5.2)$$

Interchanging the role of X and Y , we also have

$$g_M(Y, JX)g_M(H, W) = -g_M(X, Y)g_M(H, JW). \quad (5.3)$$

Thus from (5.2) and (5.3) we obtain

$$2g_M(JY, X)g_M(H, W) = 0$$

which shows that $H \in \Gamma(JD_2)$ due to the fact that g_M is a Riemannian metric. \square

Theorem 5.1. *Let F be a semi-invariant Riemannian map F from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) with totally umbilical fibers. Then either D_2 is one dimensional or the fibers are totally geodesic.*

Proof. Since the fibers are totally umbilical, we have

$$g_M(\mathcal{T}_V U, JV) = g_M(H, JV)g_M(U, V)$$

for $U, V \in \Gamma(D_2)$. Then from (3.7) we get

$$-g_M(U, \nabla_V JV) = g_M(H, JV)g_M(U, V).$$

Hence we have

$$g_M(JU, \nabla_V V) = g_M(H, JV)g_M(U, V).$$

Using (3.7) and (5.1) we obtain

$$g_M(V, V)g(H, JU) = g(U, V)g(H, JV). \quad (5.4)$$

Interchanging the role of U and V , we have

$$g_M(U, U)g(H, JV) = g(U, V)g(H, JU). \quad (5.5)$$

Then from (5.4) and (5.5) we arrive at

$$g(H, JV) = \frac{g_M(U, V)^2}{g_M(U, U)g_M(V, V)}g_M(JV, H). \quad (5.6)$$

The Eq. (5.6) implies that U and V are linearly depend or $H = 0$ due to Lemma 5.1. \square

From Theorem 5.1 and Lemma 3.2, we have the following result.

Corollary 5.1. *Let F be a semi-invariant Riemannian map F from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) with totally umbilical fibers. If $\dim(D_2) > 1$, then F is harmonic if and only if the distribution range F_* is minimal.*

Concluding remarks. A remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation $\|F_*\|^2 = \text{rank } F$ which is a link between geometrical optics and physical optics. Since the left hand side of this equation is continuous on the Riemannian manifold M and since $\text{rank } F$ is an integer valued function, this equality implies that $\text{rank } F$ is locally constant and globally constant on connected components. Thus if M is connected, the energy density $e(F) = \frac{1}{2}\|F_*\|^2$ is quantized to integer and half-integer values. The eikonal equation of geometrical optics solved by using Cauchy's method of characteristics, whereby, for real valued functions F , solutions to the partial differential equation $\|dF\|^2 = 1$ are obtained by solving the system of ordinary differential equations $x' = \text{grad } f(x)$. Since harmonic maps generalize geodesics, harmonic maps could be used to solve the generalized eikonal equation [10].

In [10], Fischer also proposed an approach to build a quantum model. He pointed out the success of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side, and Maxwell's equation, Schrödinger's equation and their proposed generalization on the physical side.

As we have seen in the introduction and the above notes, there are many applications of isometric immersions, Riemannian submersions, Riemannian maps and complex manifolds. As a generalization of anti-invariant submersions, semi-invariant Riemannian maps may have possible applications in mathematical physics.

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References

- [1] R. Abraham, J.E. Marsden, T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Springer-Verlag, 1988.
- [2] P. Baird, J.C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Clarendon Press, Oxford, 2003.
- [3] A. Bejancu, *Geometry of CR-Submanifolds*, Kluwer Academic Publishers, Dortrecht, 1986.
- [4] P. Candelas, G. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings, *Nuclear Phys. B* 258 (1985) 46–74.
- [5] B.Y. Chen, K. Ogiue, On totally real submanifolds, *Trans. Amer. Math. Soc.* 193 (1974) 257–266.
- [6] D. Chinea, Almost contact metric submersions, *Rend. Circ. Mat. Palermo* 34 (1) (1985) 89–104.
- [7] G. De Rham, Sur la réductibilité d'un espace de Riemann, *Comment. Math. Helv.* 26 (1952) 328–344.
- [8] G. Esposito, From spinor geometry to complex general relativity, *Int. J. Geom. Methods Mod. Phys.* 2 (4) (2005) 675–731.
- [9] M. Falcitelli, S. Ianus, A.M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
- [10] A.E. Fischer, Riemannian maps between Riemannian manifolds, *Contemp. Math.* 132 (1992) 331–366.
- [11] E. Garcia-Rio, D.N. Kupeli, *Semi-Riemannian Maps and their Applications*, Kluwer Academic, 1999.
- [12] A. Gray, pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.* 16 (1967) 715–737.
- [13] S. Hiepko, Eine innere Kennzeichnung der verzerrten produkte, *Math. Ann.* 241 (1979) 209–215.
- [14] S. Ianus, R. Mazzocco, G.E. Vilcu, Riemannian submersions from quaternionic manifolds, *Acta Appl. Math.* 104 (1) (2008) 83–89.
- [15] D.E. Lerner, P.D. Sommers, *Complex Manifold Techniques in Theoretical Physics*, Pitman Advanced Publishing, 1979.
- [16] J.C. Marrero, J. Rocha, Locally conformal Kähler submersions, *Geom. Dedicata* 52 (3) (1994) 271–289.
- [17] T. Nore, Second fundamental form of a map, *Ann. Mat. Pura Appl.* 146 (1987) 281–310.
- [18] K. Ogiue, Differential geometry of Kaehler submanifolds, *Adv. Math.* 13 (1974) 73–114.
- [19] B. O'Neill, The fundamental equations of a submersion, *Michigan Math. J.* 13 (1966) 459–469.
- [20] R. Penrose, Physical spacetime and nonrealizable CR-structure, *Proc. Sympos. Pure Math.* 39 (1983) 401–422.
- [21] R. Ponge, H. Reckziegel, Twisted products in pseudo Riemannian geometry, *Geom. Dedicata* 48 (1993) 15–25.
- [22] B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Cent. Eur. J. Math.* 8 (3) (2010) 437–447.
- [23] B. Şahin, Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems, *Acta Appl. Math.* 109 (2010) 829–847.
- [24] B. Şahin, Invariant and anti-invariant Riemannian maps to Kähler manifolds, *Int. J. Geom. Methods Mod. Phys.* 7 (3) (2010) 1–19.
- [25] B. Şahin, Semi-invariant Riemannian submersions from almost Hermitian manifolds, *Canadian Mathematical Bulletin*, in press (doi:10.4153/CMB-2011-144-8).
- [26] A.J. Tromba, *Teichmüller Theory in Riemannian Geometry*, Birkhauser Verlag, Boston, 1992.
- [27] B. Watson, Almost Hermitian submersion, *J. Differential Geom.* 11 (1976) 147–165.
- [28] K. Yano, M. Kon, Anti-Invariant Submanifolds, in: *Lecture Notes in Pure and Applied Mathematics*, vol. 21, Marcel Dekker, Inc., 1976.
- [29] K. Yano, M. Kon, *Structures on Manifolds*, in: *Ser. Pure Math.*, World Scientific, 1984.